Abstract Hardy Operators and their Normal Forms

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I will introduce the *normal form Hardy operator* $H_b f(x) = \int_0^{b(x)} f$, where the parameter b is a non-negative, non-increasing function on $(0, \infty)$. The *normal form Hardy inequality* $\|H_b f\|_q \le C\|f\|_p$ expresses the boundedness of H_b from unweighted $L^p(0, \infty)$ to $L^q(0, \infty)$.

I will also introduce a class of *abstract Hardy operators* and show that every *abstract Hardy inequality* is equivalent, in a strong sense, to one in normal form. This equivalence applies to Hardy operators and their duals of the weighted continuous, weighted discrete, and general measures types, as well as those based on integrals over starshaped sets in many dimensions and those involving functions defined on metric measure spaces. A straightforward formula relates each Hardy operator to the normal form parameter b of its equivalent H_b.

Besides giving a uniform treatment of many different types of Hardy operator, the reduction to normal form gives simple proofs of known theorems and a striking connection with theorems of Hardy and Bliss from 1920 and 1930. It also facilitates comparisons between Hardy operators of very different types and provides insights that lead to new results even for the most commonly studied Hardy operators.