Separable Faces and Convex Renormings of Non–separable Banach Spaces.

Some Open Problems since 1975-76, 2007 and 2020.

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Abstract

We analyze question 18 *of J. Lindenstrauss in* [4]*. We prove that a Banach space E with a norming subspace $F \subset E^*$ has an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, there is a sequence $\{A_n : \ n = 1, 2, \cdots\}$ of subsets of E such that, given any $x \in E$ and $\varepsilon > 0$, there is a $\sigma(E,F)$-open half-space H and $p \in \mathbb{N}$ such that $x \in H \cap A_p$ and the slice $H \cap A_p$ can be covered with countable many sets of diameter less than ε. Thus E has an equivalent $\sigma(E,F)$-lower semicontinuous LUR norm if, and only if, it has another one with separable denting faces,[8, 9]* **This result completely solves four problems asked in [6, Question 6.33, p.128] extending Troyanski's fundamental results (see Chapter IV in [1]), and others ones in [2, 5]. Moreover, LUR renormings are possible at points of separable faces wich could be glued as a σ-slicely isolated family of faces [6], of the unit sphere of E.** *Among new examples covered by this results are Banach spaces $C(K)$, where K is a Rosenthal compact space $K \subset \mathbb{R}^\Gamma$ i.e., a compact space of Baire one functions on a Polish space Γ, with at most countably many discontinuity points for every $s \in K$,* **which solves three problems asked in [6, Question 6.23, p.125].** *Previously, it was only known for K being separable too, see [3] where the σ-fragmentability of $C(K)$ was already proved for non separable K, and a conjecture for the pointwise lower semicontinuous and LUR renorming presented here was posed, details will appear in [7].*

For strictly convex renormings **we solve a recent question of R. Smith [11] giving a final answer to Lindenstauss question 18 in [4], see [9] and [10].** *Indeed, we prove that E admits an equivalent $\sigma(E,F)$-lower semicontinuous and strictly convex norm if, and only if, it has another one with separable faces. A purely topological new characterization follows for dual spaces and dual norms.*

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