CHARACTERIZED SUBGROUPS OF THE CIRCLE GROUP

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A subgroup H of the circle group $\mathbb{T}$ is said to be *characterized* by a sequence of integers (u_n), if $H = \{x \in \mathbb{T} : u_n x \to 0\}$. This notions can be appropriately defined also in arbitrary topological groups. It is deeply rooted in Number Theory ([3, 14]), Topological Algebra ([12]), Harmonic Analysis ([1]) and Descriptive Set Theory ([2, 7]).

The talk will focus on the following issues.

1) Historical background on the origin of this field.

2) Basic properties of the characterized subgroups. These subgroups are Borel (hence, measurable) sets, their size depends on the asymptotic behavior of the characterizing sequences (u_n) (e.g., Eggleston dychotomy [13]).

3) When a subgroup of $\mathbb{T}$ is characterizable? All countable subgroups of $\mathbb{T}$ are characterizable [5, 3], this was extended to arbitrary compact metrizable abelian groups [4, 10] (but fails to be true in the non-metrizable case [11]). On the other hand, F_σ-subgroups of $\mathbb{T}$ need not be characterizble [2].

4) A new trend. Replacing the usual convergence by statistical convergence, statistically characterized subgroups were introduced recently in [6]. Essential role in the definiion of statistical convergence plays the ideal $\mathcal{I}_d$ of subsets of $\mathbb{N}$ of asymptotic density 0. Further step in this direction was done in [9], where the ideal $\mathcal{I}_d$ was replaced by an arbitrary ideal $\mathcal{I}$ of $\mathbb{N}$ and *ideal convergence* (in the sense of Cartan [8]) was used. (A sequence (x_n) in a topological space X is said to *$\mathcal{I}$-converge* to a point $x \in X$, if $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ for every neighborhood U of x in X.)

We discuss the counterparts of the properties from 2) in this more general context, focusing on the impact of the properties of the ideal $\mathcal{I}$ in this apsect.

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