THE CLASSICAL OPERATORS ON THE SPACE OF REAL ANALYTIC FUNCTIONS

The aim of this thesis is to study operators acting on the space of real analytic functions $\mathscr{A}(\mathbb{R})$. We focus on three classes of operators: Hadamard multiplier operators, Hankel operators and Toeplitz operators.

The study of the Hadamard multipliers concentrates on the problem of generating strongly continuous semigroup by these operators. Based on the theory developed by P. Domański and M. Langenbruch, we give a generation theorem for Hadamard multipliers on the space $\mathscr{A}(\mathbb{R})$. Next, we apply it to the classical examples of multipliers. The first ones are finite order Euler differential operators, $E = \sum_{n=0}^{N} a_n \theta^n$, $\theta f(x) = xf'(x)$, $a_0, \ldots, a_n \in \mathbb{C}$. We prove that the first order Euler differential operator $E = a\theta + bI$ generates a strongly continuous semigroup if and only if $a \in \mathbb{R}$. Next we show the other cases when finite order Euler differential operator is not a generator of a strongly continuous semigroup. Unfortunately, we were not able to obtain a full characterization of Euler differtial generators. The other multiplier which we consider is the Hardy averaging operator $Hf(x) = \frac{1}{x}\int_0^\infty f(y)dy$. We prove that every operator of the form $M = \sum_{n=0}^{N} a_n H^n$, $a_0, \ldots, a_n \in \mathbb{C}$, generates a strongly continuous semigroup on $\mathscr{A}(\mathbb{R})$.

Next, we study the Hankel operators. We give the integral representation of Hankel operators and prove that the space of all Hankel operators on $\mathscr{A}(\mathbb{R})$ with the topology induced from $L_b(\mathscr{A}(\mathbb{R}))$ is topologically isomorphic to the Fréchet space of entire functions $H(\mathbb{C})$. We also investigate the spectra and other properties of Hankel operators on $\mathscr{A}(\mathbb{R})$. We prove that the spectrum of Hankel operator $H\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is equal to the point spectrum and the sequence of its eigenvalues belongs to the space of rapidly decreasing sequences s.

Finally, we study Toeplitz operators acting on $\mathscr{A}(\mathbb{R})$. P. Domański and M. Jasiczak proved that a Toeplitz operator on $\mathscr{A}(\mathbb{R})$, similarly to the classical one, is a compression of a multiplication operator. More precisely they proved that an operator $T\colon \mathscr{A}(\mathbb{R}) \to \mathscr{A}(\mathbb{R})$ is a Toeplitz operator if and only if there exists a function $F \in \mathcal{X}$ such that $T = \mathcal{C}M_F$. The symbol space $\mathcal{X}$ is defined as the inductive limit of Fréchet spaces:

$$\mathcal{X} = \operatorname{ind}_{U,K} H(U \setminus K)$$

where U runs over all open complex neighbourhoods of $\mathbb{R}$ and K runs through all compact sets of $\mathbb{R}$. The symbol $\mathcal{C}$ denotes the appropriate Cauchy transform which is also a projection from $\mathcal{X}$ onto $\mathscr{A}(\mathbb{R})$. The operator M_F is the operator of multiplication by F. We give a characterization of left-sided invertible Toeplitz operators, which together with the result of M. Jasiczak on right-side invertibility solves the problem of one-sided invertibility of Toeplitz operators.

A commutator of two operators A and B is given by $[A, B] := AB - BA$. We give a complete characterization of the finite rank commutators of Teoplitz operators on $\mathscr{A}(\mathbb{R})$.

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