Turán and Ramsey numbers for 3-uniform hyperpaths

Eliza Jackowska-Boryc

At the beginning of my thesis I introduce basic terminology and definitions. For $k \geqslant 2$, a *k-uniform hypergraph* (or *k-graph*, for short) is an ordered pair $H = (V, E)$, where V is a finite, non-empty set of vertices and $E \subseteq \binom{V}{k}$ is a set of distinct, k-element subsets of V, called edges.

Major questions in the extremal hypergraph theory are related to determining Turán numbers and Ramsey numbers. Determining Ramsey numbers is one of the most difficult problems related to Ramsey's theorem (Graham, Rothschild, Spencer, 1980). In this dissertation we will focus on hypergraph Ramsey numbers. Classical Ramsey numbers are hard, and very little is known. For this reason, research in hypergraph Ramsey theory was directed to other structures with less density, for instance, hyperpaths or hypercycles.

There are several definitions of k-uniform paths and cycles. In this thesis we focus on the symmetric case when the intersections of consecutive edges have a fixed size equal to 1. *A loose path* P_l^k is a k-graph with l edges $e_1, ..., e_l$ such that $|e_i \cap e_j| = 0$ if $|i - j| > 1$ and $|e_i \cap e_j| = 1$ if $|i - j| = 1$. In my thesis, I focus mainly on 3-uniform loose path of length 3, P_3^3. As a helpful tool, in our proofs we often use a 3-uniform cycle of length 3, C_3^3, which is called the triangle.

Our main contribution to Ramsey Theory is Theorem 2.2, which asserts that $R(P_3^3; r) = r+6$, for the number of colors $r \leqslant 7$, while Theorem 2.1, is a particular case of Theorem 2.2 for $r = 3$. Those theorems are proved in Chapter 3. The proof of the lower bound is based on a construction given by Gyárfás and Raeisi (2012). My version of that construction yields that for $r \geqslant 2$, if a k-graph F is not a star then $R(F; r) \geqslant r + |V(F)| - 1$. We conduct two proofs of the upper bound in Theorem 2.1. The first proof is unpublished, self-contained, and relies on a detailed case analysis, while the second proof is based on a general strategy of finding upper bounds on Ramsey numbers via Turán numbers.

For a family of k-graphs $\mathcal{F}$ and a positive integer n, the *Turán number* $\mathrm{ex}_k(n; \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free k-graph on n vertices. An n-vertex k-graph H is called *extremal* with respect to $\mathcal{F}$ if H is $\mathcal{F}$-free and $|H| = \mathrm{ex}_k(n; \mathcal{F})$. We denote by $\mathrm{Ex}_k(n; \mathcal{F})$ the set of all pairwise non-isomorphic n-vertex k-graphs which are extremal with respect to $\mathcal{F}$.

The Turán number $\mathrm{ex}_3(n; P_3^3)$ is determined for all n in Theorem 2.7. Nevertheless, it turns out that the main tool to determine multicolor Ramsey numbers is another variant of Turán numbers with an iterative definition. For a k-graph F and integers $s, n \geqslant 1$, the Turán number of the s-th order is defined as $\mathrm{ex}_k^{(s)}(n; F) = max\{|E(H)| : |V(H)| = n, H \not\supseteq F$, and $\forall H' \in Ex_k^{(1)}(n; F) \cup \cdots \cup Ex_k^{(s-1)}(n; F), H \not\subseteq H'\}$, if such k-graph H does exist. An n-vertex k-graph H where $H \not\supseteq F$ is called s-extremal for F if $|E(H)| = ex_k^{(s)}(n; F)$ and $\forall H' \in Ex_k^{(1)}(n; F) \cup \cdots \cup Ex_k^{(s-1)}(n; F), H \not\supseteq H'$. We denote by $Ex_k^{(s)}(n; F)$ the family of n-vertex k-graphs which are s-extremal for F.

In order to prove Theorem 2.2 we have to determine the second order Turán number $\mathrm{ex}_3^{(2)}(n; P_3^3)$ (Theorem 2.9) and the third order Turán number $\mathrm{ex}_3^{(3)}(12; P_3^3)$ (Theorem 2.10). We finish Chapter 3 with the proof of Theorem 2.2, emphasizing the necessity of applying the Turán numbers of higher orders.

Then, we focus on the proof of Theorem 2.7, which determines the Turán number $\mathrm{ex}_3(n; P_3^3)$ and assigns a unique extremal graph for each n. The main idea of our proof is to link the presence of a copy of P_3^3 with the presence of the triangle C_3^3. This is why at the beginning of Chapter 4 we need to introduce another variation on Turán numbers. For a k-graph F, and another k-graph G where $F \nsubseteq G$, and an integer $n \geqslant |V(G)|$, the conditional Turán number is defined as $\mathrm{ex}_k(n; F|G) = max\{|E(H)| : |V(H)| = n, H \nsupseteq F, \text{ and } H \supseteq G\}$.

We present two proofs of Theorem 2.7. The first was published in our paper (Jackowska, Polcyn, Ruciński, 2016), while the idea of the second proof of Theorem 2.7 was suggested by one of the reviewers of that paper (see acknowledgements therein).

Conditional Turán numbers are needed not only in the proof of Theorem 2.7, but also later in Chapter 5. We finish Chapter 4 with a consideration of some additional problems related to conditional Turán numbers for non-intersecting 3-graphs and state two important results, Theorem 4.9 and Theorem 4.11, which are needed in the proofs of Theorems 2.9 and 2.10.

In Chapter 5 we focus on the proofs of Theorem 2.9, determining the second order Turán number $\mathrm{ex}_3^{(2)}(n; P_3^3)$ for all n, and Theorem 2.10, determining the third order Turán number $\mathrm{ex}_3^{(3)}(12; P_3^3)$. In their proofs we use some results stated earlier, most notably Theorem 4.9.

We finish our dissertation with a short summary of the main results and suggest some related open problems.

Eliza Jackowska-Boryc