Some remarks on the isomorphic classification of spaces of Lipschitz functions and Lipschitz-free spaces

Since the spaces $\mathrm{Lip}_0(M)$ are somewhat close to L^∞-spaces, one might wonder whether they are all essentially the same. It turns out that the question of isomorphic classification of spaces of Lipschitz functions and even of Lipschitz-free spaces is an immensely difficult one. One of the most famous results is that the spaces $\mathrm{Lip}_0(\mathbb{R})$ and $\mathrm{Lip}_0(\mathbb{R}^2)$ and therefore also $\mathcal{F}(\mathbb{R})$ and $\mathcal{F}(\mathbb{R}^2)$ are not isomorphic. It is however still unknown whether $\mathcal{F}(\mathbb{R}^2)$ and $\mathcal{F}(\mathbb{R}^3)$ are isomorphic.

In addition to asking whether the spaces $\mathcal{F}(M)$ and $\mathcal{F}(N)$ or $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_0(N)$ for metric spaces M and N are isomorphic or even isometric, one might also ask the question of whether an isomorphism between these spaces can be constructed explicitly.

We discuss some of the history of the quest for the isomorphic (and isometric) classification of the spaces $\mathrm{Lip}_0(M)$ of Lipschitz functions and of the Lipschitz-free spaces $\mathcal{F}(M)$ and some recent contributions to this question.