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A summary of papers:

I. D. Bugajewska, D. Bugajewski, P. Kasprzak, P. Maćkowiak, *Nonautonomous superposition operators in the spaces of functions of bounded variation*, Topological Method in Nonlinear Analysis 48 (2016), 637–660.

II. P. Kasprzak, P. Maćkowiak, *Local boundedness of nonautonomous superposition operators in* $BV[0,1]$, Bulletin of the Australian Mathematical Society 92 (2015), 325–341.

III. P. Maćkowiak, *On the continuity of superposition operators in the space of functions of bounded variation*, Aequationes Mathematicae, doi:10.1007/s00010-017-0491-x (2017), 1–19.

Contents

1 Introduction

The notion of variation, introduced by C. Jordan in 1881 (see [21]), is one of the basic notions of mathematical analysis. Since the end of the 19th century the Jordan variation as well as its generalizations and extensions have been an object of interest of many mathematicians due to the fact that functions of bounded variation in the sense of Jordan have found applications in many fields, for example in the geometric measure theory (see e.g. [1, 17, 25]), in the theory of Fourier series (see [31]), in the theory of integration and integral equations (see [7, 8, 14]), in image processing, analysis and recovery (see e.g. [10–12, 18, 23, 30]), and in economics (see [19]). Many of those applications involve nonlinear superposition operators defined on the space of functions of bounded variations in the sense of Jordan. It stems from the fact that the theory of nonlinear superposition operators in the spaces of functions of bounded variation in the sense of Jordan is closely connected with the examination of solutions to nonlinear equations in these classes of functions (see e.g. [8, 13, 20]). The study of such solutions seems to be interesting for at least a few reasons. First, let us draw attention to the fact that solutions to the classical Cauchy problem for the equation of first order, defined on a compact interval in $\mathbb{R}$, the existence of which is guaranteed by the classical Peano theorem, are functions of bounded variation in the sense of Jordan (at least locally). This property is preserved, if one considers solutions to this equation the existence of which follows from the classical Carathéodory theorem (see [15, Theorem 1.1]). Second, solutions to many equations which describe concrete physical phenomena are functions of (local) bounded variation. As examples we could mention here equations describing the amplitude of forced vibrations of a string, which appear in engineering (see [29]), or Volterra integral equations modeling population dynamics under constant harvesting (see e.g. [5]). The motivation for the study of solutions to nonlinear integral equations in the class of functions of bounded variation in the sense of Jordan comes also from the theory of non-absolute convergent integrals. Namely, it is well-known that if $h\colon [0,1] \to \mathbb{R}$ is a function integrable in the Denjoy–Perron sense (or, equivalently, in the Henstock-Kurzweil sense), then $h\varphi$ is also integrable in that sense whenever $\varphi\colon [0,1] \to \mathbb{R}$ is a function of bounded variation in the sense of Jordan (see [14]).

In the recently published monograph [2] which aims at giving a thorough account of functions of bounded (generalized) variation, their relation to other important classes of functions as well as their applications to various problems arising in nonlinear analysis, the authors stated three basic open problems regarding nonautonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. The first and the most fundamental problem mentioned there concerns both necessary and sufficient conditions which would imply that the nonautonomous superposition operator maps the space under consideration into itself. The second problem is whether a nonautonomous superposition mapping the space bounded variation in the sense of Jordan into itself is automatically bounded, that is if it maps bounded subsets of that space into its bounded subsets. The third problem concerns the continuity of the nonlinear superposition operator acting in the space under consideration.

Thus we can say, roughly speaking, that the theory of nonautonomous superposition operators in the space of functions of bounded variation in the sense of Jordan was in its initial point according to the monograph [2]. The main reason for writing the papers I-III was to build a firm base for the theory and applications of nonlinear superposition operators in the space of functions of bounded variation in the sense of Jordan by solving the three just mentioned problems.

1.1 Preliminaries

Let us recall some basic, notions and properties we shall use in the text. For a function $u : [a, b] \to \mathbb{R}$, $a < b$, its (Jordan) variation is defined by the formula

$$\bigvee_a^b u := \sup\{\sum_{i=1}^k |u(t_i) - u(t_{i-1})| : a = t_0 < t_1 < \ldots < t_k = b,\ k \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of positive integers. If $\bigvee_a^b u < +\infty$, then we say that u is of bounded variation (in the sense of Jordan). The set of all functions of bounded variation in the sense of Jordan defined on $[a, b]$ endowed with the norm

$$||u||_{BV} := |u(a)| + \bigvee_a^b u,$$

is a Banach space (see [2, p. 62]); we denote that space by $BV[a, b]$ and put $BV := BV[0, 1]$. If $f : \mathbb{R} \to \mathbb{R}$, then we define the autonomous superposition operator $F : BV \to BV$, generated by f, by $F(u)(t) := f(u(t))$, $u \in BV$, $t \in [0, 1]$. If $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$, then the nonautonomous superposition operator (or the Nemytskii operator) $F : BV \to BV$, generated by f, is defined by the equality $F(u)(t) := f(t, u(t))$, $u \in BV$, $t \in [0, 1]$; the function f is said to be the generator of F. Let us observe that the notion of nonautonomous superposition operator is obviously more general than that of the autonomous superposition operator.

We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol $\lfloor a \rfloor$ stands for the greatest integer number not exceeding a, $a \in \mathbb{R}$. We say that a function f is L-lipschitzian if f is lipschitzian with a Lipschitz constant L. We take on the convention $\sum_{i \in \emptyset} := 0$. By $C^1((a, b) \times \mathbb{R}, \mathbb{R})$, $a < b$, $a, b \in \mathbb{R}$, we denote the space of continuously differentiable functions from the product $(a, b) \times \mathbb{R}$ to $\mathbb{R}$. For $t \in [0, 1]$ and $\varepsilon \in (0, +\infty)$ we will write $l_\varepsilon(t) := \max\{0, t - \varepsilon\}$ and $r_\varepsilon(t) := \min\{1, t + \varepsilon\}$. Finally, $\theta(t) := 0$, $t \in [0, 1]$.

The closed ball with center at x and radius $r \in (0, +\infty)$ in a normed space X is denoted by $B_X(x, r)$. For simplicity, instead of $B_\mathbb{R}(x, r)$ we will obviously write $[x - r, x + r]$.

A mapping G: $BV \to BV$ is said to be locally bounded if for each $r > 0$ there exists $R > 0$ such that $G(B_{BV}(0, r)) \subset B_{BV}(0, R)$.

1.2 Some remarks about this summary

In this summary we present the most significant results contained in the papers I-III, so some results have been omitted. The order of appearance or numbering of theorems, lemmas, corollaries, etc., may differ from their numbering in the considered source papers; here we assume the continuous numeration. Moreover, we unified notations in the summary, so the form (but not the content) of the presented results might slightly differ from their original versions. Unless otherwise stated, results presented in the summary originate from the papers I-III. In Sections 2-4, whenever we write 'bounded variation', we mean 'bounded variation in the sense of Jordan'.

1.3 My contribution to the papers I, II

My participation in the work on the paper I consisted in elaborating the proof of Theorem 11 below (Theorem 3.8 in I) and the development of Section 2.2 below (Section 4 in I). The case of locally bounded functions (except for last proof presented in this section). I assess my contribution to the paper I as of 30%.

My participation in the work on the paper II consisted in elaborating the line of the proof of the main result of the work, Theorem 31 below (Theorem 4.1 in II), including preliminary form of lemmas preceding the theorem and their proofs. I assess my contribution to the paper as of 70%.

2 Acting conditions: the paper I

In the monograph [4], on page 175 the Authors write:

> As already mentioned, no general results on the acting, boundedness, or continuity of the superposition operator F are known in the nonautonomous case $f = f(t, u)$ (apart from trivial sufficient conditions, of course).

On page 174 of that monograph the Authors quote and prove the following result coming originally from Ljamin's paper [24].

Theorem 1. *Assume that the function $f(t, \cdot)$ satisfies the Lipschitz condition on $\mathbb{R}$ uniformly in $t \in [0, 1]$, and that the function $f(\cdot, u)$ is of bounded variation on the interval $[0, 1]$, uniformly in $u \in \mathbb{R}$. Then the nonautonomous superposition operator F, generated by f, maps the space BV into itself and is locally bounded, that is, it maps bounded sets into bounded ones.*

In the paper [6], D. Bugajewska formulated the conjecture that Theorem 1 might not be true. Let us also add that the proof of Ljamin's theorem presented in the survey article [3] is false. One can find the suitable examples confirming its falsity in the review by D. Bugajewski for ZblMATH (Zbl 1255.47059). The conjecture from the paper [6] was confirmed, for example, in my paper [26] with help of the following counterexample

Example 2 ([26]). Let the function $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(t, u) = \begin{cases} 0, & \forall\, n \in \{2, 3, \ldots\} : t \neq c_n \text{ or } u \notin I_n, \\ \dfrac{1}{n}\left(1 - \dfrac{|u - c_n|}{w_n}\right), & \exists\, n \in \{2, 3, \ldots\} : t = c_n \text{ and } u \in I_n, \end{cases}$$

where $c_n = 1 - \frac{1}{n}$, $w_n = \frac{1}{2n}$ and $I_n = (c_n - w_n, c_n + w_n)$ for $n = 2, 3, \ldots$. For an arbitrary $t \in [0, 1]$, the function $f(t, \cdot)$ satisfies the Lipschitz condition (uniformly in the second variable) with a Lipschitz constant not greater than 2. Moreover, $\bigvee_0^1 f(\cdot, u) \leq 22$ for an arbitrary $u \in \mathbb{R}$. However, considering the functions $x(t) = t$ and $g(t) = f(t, x(t))$ for $t \in [0, 1]$, one can easily check that the nonautonomous superposition operator, generated by the function f, does not map the space BV into itself.

For the case of nonautonomous superposition operators in the space of functions of bounded variation, the sufficient acting conditions are given in the following

Theorem 3 ([6, Theorem 1]). *Let a function $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:*

(i) *f satisfies a Lipschitz condition on $\mathbb{R}$ uniformly in $t \in [0, 1]$;*

(ii) *there exists a constant $M > 0$ such that for arbitrary real numbers $u_0, u_1, \ldots, u_{n-1}$ and an arbitrary finite partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$, the following inequality holds*

$$\sum_{i=1}^{n} |f(t_i, u_{i-1}) - f(t_{i-1}, u_{i-1})| \leq M. \tag{1}$$

Then the nonautonomous superposition operator F, generated by f, maps the space BV into itself and is locally bounded.

The above quoted result gave us clue what kind of conditions might have been necessary in the situation under consideration.

2.1 Nonautonomous superposition operators - a general case

The first result of this section is a simple refinement of Theorem 3.

Theorem 4. *Let the function $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:*

(i) *f satisfies a local Lipschitz condition on $\mathbb{R}$, uniformly in $t\in[0,1]$;*

(ii) *for every $r>0$ there exists a constant $M_r>0$ such that for every $n\in\mathbb{N}$, every partition $0=t_0<\ldots<t_n=1$ of $[0,1]$ and every $u_0,\ldots,u_{n-1}\in[-r,r]$, the following implication holds*

$$\sum_{i=1}^{n-1}|u_i-u_{i-1}|\le r \qquad\Longrightarrow\qquad \sum_{i=1}^{n}|f(t_i,u_{i-1})-f(t_{i-1},u_{i-1})|\le M_r. \tag{2}$$

Then the superposition operator F, generated by f, maps the space BV into itself and is locally bounded.

The following example shows that the above result is an essential improvement of Theorem 3.

Example 5. Let us consider the function $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ defined by the following formula

$$f(t,u)=\begin{cases}0, & \text{if } t\ne\frac{1}{n} \text{ and } u\in\mathbb{R},\\ 0, & \text{if } t=\frac{1}{n} \text{ and } u<n-1,\\ 1, & \text{if } t=\frac{1}{n} \text{ and } u\ge n,\\ u-(n-1), & \text{if } t=\frac{1}{n} \text{ and } n-1\le u<n,\end{cases}$$

where $n\in\mathbb{N}$.

Let us observe that for any $t\in[0,1]$ the function $u\mapsto f(t,u)$ satisfies a Lipschitz condition with the constant 1. Furthermore, f does not satisfy the condition (ii) of Theorem 3. Indeed, for every positive integer $n\ge 2$ let

$$u_0{:=}0,\qquad u_1{:=}n-1,\qquad \ldots,\qquad u_i{:=}n-i,\qquad \ldots,\qquad u_{n-1}{:=}1$$

and

$$t_0{:=}0,\qquad t_1=\frac{1}{n},\qquad \ldots,\qquad t_i{:=}\frac{1}{n-i+1},\qquad \ldots,\qquad t_n{:=}1.$$

Then

$$\sum_{i=1}^{n}|f(t_i,u_{i-1})-f(t_{i-1},u_{i-1})|$$
$$=\left|f(\tfrac{1}{n},0)-f(0,0)\right|+\sum_{i=2}^{n}\left|f(\tfrac{1}{n-i+1},n-i+1)-f(\tfrac{1}{n-i+2},n-i+1)\right|\ge n-1.$$

On the other hand, the function f satisfies the condition (ii) of Theorem 4, since for an arbitrary positive number $r > 0$, in view of the fact that in each rectangle $[0,1] \times [-r,r]$, the function f vanishes except at the set consisting of a finite number of vertical line segments, it suffices to take $M_r{:=}2\lfloor r \rfloor + 1$.

Proposition 6. *Suppose that the function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the assumption* (i) *of Theorem* 4. *If the autonomous superposition operator F, generated by f, maps the space BV into itself and is locally bounded, then the function f satisfies the condition* (ii) *of Theorem* 4.

The next example and Propositions 8-9 characterize the situation, where the generator f is not a locally bounded function, that is it maps a bounded set onto an unbounded set.

Example 7. Let F be the nonautonomous superposition operator generated by a function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$. Let us emphasize that the fact that F maps the space BV into itself does not have to imply that F is locally bounded.

Indeed, let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f(t,u) = \begin{cases} 0, & \text{if } t \neq 0 \text{ or } u \leq 0, \\ u^{-1}, & \text{otherwise.} \end{cases}$$

Furthermore, for every $n \in \mathbb{N}$, let

$$x_n(t) = \begin{cases} n^{-1}, & \text{if } t = 0, \\ 0, & \text{if } t \in (0,1]. \end{cases}$$

Clearly, the superposition operator F, generated by the function f, maps BV into itself. However, $\|F(x_n)\|_{BV} = 2n$, while $\|x_n\|_{BV} = 2n^{-1}$ for $n \in \mathbb{N}$.

Actually, the conclusion of the above remark follows from a more general result, namely from the following

Proposition 8. *Suppose that a function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ generates the nonautonomous superposition operator F which maps the space BV into itself. If the function f is not locally bounded, then also the operator F is not locally bounded.*

The fact that nonautonomous superposition operator maps the space BV into itself implies also the property stated as

Proposition 9. *If the nonautonomous superposition operator F, generated by a function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$, maps the space BV into itself, then for every $r > 0$ the set $T_r{:=}\{t \in [0,1] : \sup_{u \in [-r,r]} |f(t,u)| = +\infty\}$ is finite.*

The following result states, in particular, that thinking about necessary acting conditions for the nonautonomous superposition in the space BV, one cannot say anything about the behavior of the generator of that operator with respect to the second variable.

Theorem 10. *Let F be a nonautonomous superposition operator, generated by a function $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$, which maps the space BV into itself. Then for every $u\in\mathbb{R}$, the function $t\mapsto f(t,u)$ is of bounded variation. Furthermore, in general, nothing can be said about the function $u\mapsto f(t,u)$, where $t\in[0,1]$ is fixed.*

The main result concerning necessary and sufficient conditions for the inclusion $F(BV)\subset BV$ and the local boundedness of the nonautonomous superposition operator F is the following

Theorem 11. *Suppose that $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a given function. The following conditions are equivalent:*

(i) *the nonautonomous superposition operator F, generated by f, maps the space BV into itself and is locally bounded;*

(ii) *for every $r>0$ there exists a constant $M_r>0$ such that for every $k\in\mathbb{N}$, every finite partition $0=t_0<\ldots<t_k=1$ of the interval $[0,1]$ and every finite sequence $u_0,u_1,\ldots,u_k\in[-r,r]$ with $\sum_{i=1}^{k}|u_i-u_{i-1}|\leq r$, the following inequalities hold*

$$\sum_{i=1}^{k}\left|f(t_i,u_i)-f(t_{i-1},u_i)\right|\leq M_r \quad \textit{and} \quad \sum_{i=1}^{k}\left|f(t_{i-1},u_i)-f(t_{i-1},u_{i-1})\right|\leq M_r.$$

Remark 12. Let us add that Theorem 4 and Proposition 6 can be obtained as corollaries to Theorem 11.

2.2 The case of locally bounded functions

In this subsection, unless stated otherwise, we assume that $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ maps bounded sets into bounded sets and that F is the nonautonomous superposition operator generated by f.

Reasoning similar as in the proof of [32, Lemma 1] leads to

Lemma 13. *Let $x\colon [0,1]\to\mathbb{R}$. Then $F(x)\notin BV$ if and only if there exists $t\in[0,1]$ such that*

$$\bigvee_{l_\alpha(t)}^{r_\alpha(t)} F(x)=+\infty \qquad \textit{for every } \alpha>0. \tag{3}$$

Applying Lemma 13 we proved the following technical result, which was crucial for our further considerations.

Lemma 14. *Suppose that there exists $x\in B_{BV}(0,r)$, where $r>0$, such that $F(x)\notin BV$ and let $t\in[0,1]$ satisfy the condition (3). Then for every $\delta>0$ there exists $u\in[-r,r]$ such that for every $q\in\mathbb{N}$ there exist positive integers c_q,d_q and a finite collection of points $l_{1/c_q}(t)\leq t_0^q<t_1^q<\ldots<t_{d_q}^q\leq r_{1/c_q}(t)$ such that the following properties hold: $x(t_i^q)\in[u-\delta,u+\delta]$ for $i=0,1,\ldots,d_q$, $c_q\to+\infty$ as $q\to+\infty$ and*

$$\lim_{q\to\infty}\sum_{i=1}^{d_q}\left|f(t_i^q,x(t_i^q))-f(t_{i-1}^q,x(t_{i-1}^q))\right|=+\infty. \tag{4}$$

From Lemma 14 we derived the following

Lemma 15. *Let the function $x \in B_{BV}(0,r)$, where $r > 0$, be such that $F(x) \notin BV$ and let $t \in [0,1]$ satisfy the condition (3). Then, there exists $u \in [-r,r]$ such that for every $\varepsilon > 0$ and every sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers convergent to 0 there exist sequences of positive integers $k_n{:=}k(\delta_n,\varepsilon)$ and finite collections of points $l_\varepsilon(t) \le t_0^{\delta_n,\varepsilon} < t_1^{\delta_n,\varepsilon} < \ldots < t_{k_n}^{\delta_n,\varepsilon} \le r_\varepsilon(t)$ for which the following conditions hold: $x(t_i^{\delta_n,\varepsilon}) \in [u-\delta_n, u+\delta_n]$, for $i = 1,\ldots,k_n$ and*

$$\lim_{n\to\infty}\sum_{i=1}^{k_n}\left|f(t_i^{\delta_n,\varepsilon}, x(t_i^{\delta_n,\varepsilon})) - f(t_{i-1}^{\delta_n,\varepsilon}, x(t_{i-1}^{\delta_n,\varepsilon}))\right| = +\infty.$$

Is is clear that if for a function $x \in B_{BV}(0,r)$, where $r > 0$, there exist points $t \in [0,1]$ and $u \in [-r,r]$ such that the claim of Lemma 15 is satisfied, then $F(x) \notin BV$. Therefore, we could state

Theorem 16. *Let $x \in B_{BV}(0,r)$ for some $r > 0$. Then $F(x) \in BV$ if and only if for any $t \in [0,1]$ and $u \in [-r,r]$ there exist $\varepsilon > 0$, $\delta > 0$ and $M > 0$ such that $l_\varepsilon(t) \le t_0 < t_1 < \ldots < t_k \le r_\varepsilon(t)$ and $x(t_i) \in [u-\delta, u+\delta]$ for $i \in \{0,1,\ldots,k\}$, $k \in \mathbb{N}$, imply*

$$\sum_{i=1}^{k}\left|f(t_i, x(t_i)) - f(t_{i-1}, x(t_{i-1}))\right| < M. \tag{5}$$

We can restate Theorem 16 in terms of the properties of the generator function, but we need the following

Definition 17. (a) Let $A \subset \mathbb{R}$ be a non-empty set and let $a, b \in \mathbb{R}$ be such that $a < b$. A finite sequence $(t_i, u_i)_{i=1}^k$ is called a *flagged partition* of $[a,b] \times A$, if $a \le t_0 < t_1 < \ldots < t_k \le b$ and $u_i \in A$ for $i = 1,\ldots,k$.

(b) If V^1 and V^2 are flagged partitions of $[a,b] \times A$, then V^2 is called a *condensation* of V^1 (which we will denote by $V^1 \preceq V^2$), if V^1 is a subsequence of V^2.

(c) A sequence $(V^n)_{n\in\mathbb{N}}$ of flagged partitions of $[a,b] \times A$ is called a *condensation sequence* of $[a,b] \times A$, if $V^n \preceq V^{n+1}$ for $n \in \mathbb{N}$.

(d) A condensation sequence $(V^n)_{n\in\mathbb{N}}$ of $[a,b] \times A$, where $V^n = (t_i^n, u_i^n)_{i=0}^{k_n}$, is called *proper*, if $\sup_n \sum_{i=0}^{k_n}|u_i^n - u_{i-1}^n| < +\infty$.

Theorem 18. *The operator F maps BV into itself if and only if for any $t \in [0,1]$ and $u \in \mathbb{R}$ there exist $\varepsilon > 0$ and $\delta > 0$ such that for any proper condensation sequence $(V^n)_{n\in\mathbb{N}}$ of $[l_\varepsilon(t), r_\varepsilon(t)] \times [u-\delta, u+\delta]$, where $V^n = (t_i^n, u_i^n)_{i=0}^{k_n}$, it holds*

$$\sup_n \sum_{i=1}^{k_n}\left|f(t_i^n, u_i^n) - f(t_{i-1}^n, u_{i-1}^n)\right| < +\infty.$$

2.3 Nonautonomous superposition operators - a separable variables case

In this section we are going to study nonlinear superposition operators which are generated by functions with separable variables, that is, functions of the form $(t,u) \mapsto f(t)g(u)$, where $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$. Although the results of this section may be treated as corollaries to Theorem 11, however their proofs - due to a specific form of generator - can be significantly simplified.

The following simple result explains when a nonautonomous superposition operator acting in a function space is generated by a function with separable variables.

Theorem 19. *Let X be a vector space over $\mathbb{R}$ satisfying the following conditions:*

(i) *X is a vector subspace of the vector space all real-valued functions defined on $[0,1]$ considered with the standard pointwise addition and multiplication by scalars;*

(ii) *X contains all constant functions.*

Moreover, assume that F is a nonautonomous superposition operator which maps the vector space X into itself. The superposition operator F is generated by a function of the form $(t,u) \mapsto f(t)g(u)$, where $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ if and only if there exists $u_0 \in \mathbb{R}$ such that for every $u \in \mathbb{R}$ there is $a_u \in \mathbb{R}$ such that $F(x_u) = a_u F(x_{u_0})$, where x_u denotes the constant function $x(t) = u$, $t \in [0,1]$ (similarly for x_{u_0}) . Furthermore, if $F \not\equiv 0$, then the functions f and g are uniquely determined up to a constant factors α and β, respectively, such that $\alpha\beta = 1$.

Clearly, if $f \equiv 0$, then from the fact that the superposition operator F, generated by the function $(t,u) \mapsto f(t)g(u)$, maps the space X into itself, nothing can be inferred about properties of g. However, the more is known about the behavior of the function f at points of its continuity, the more can be proved about g.

Now, let us pass on to the space BV.

Theorem 20. *Let $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions satisfying the following conditions:*

(i) *there exists $u_0 \in \mathbb{R}$ such that $g(u_0) \neq 0$;*

(ii) *there exists a point $t_0 \in [0,1]$ of continuity of f such that $f(t_0) \neq 0$.*

Then the nonautonomous superposition operator F generated by the function $(t,u) \mapsto f(t)g(u)$ maps the space BV into itself if and only if:

(a) *$f \in BV$;*

(b) *g satisfies a local Lipschitz condition.*

Remark 21. Let us observe that the conditions (a) and (b) of Theorem 20 guarantee that the superposition operator F generated by the function $(t,u) \mapsto f(t)g(u)$ is locally bounded. Therefore, in that case, similarly to the autonomous case, the fact that the superposition operator F maps the Banach space BV into itself implies its local boundedness.

Now, let us consider the case when the function f vanishes at each point of continuity.

Theorem 22. *Let $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions, and assume that*

(i) *$f(t) = 0$ at every point $t \in [0,1]$ of continuity of f.*

The nonautonomous superposition operator F generated by the function $(t,u) \mapsto f(t)g(u)$ maps the space BV into itself and is locally bounded, whenever

(ii) *$f \in BV$;*

(iii) *g is locally bounded.*

Remark 23. If $f \in BV$, then the assumption (i) of Theorem 22 implies that $f(t) \neq 0$ if and only if $t \in D_f$, where D_f is the set of discontinuity points of f.

Let us observe that if we drop the assumption that f is of bounded variation, then the claim that the assumption (i) of Theorem 22 implies that $f(t) \neq 0$ if and only if $t \in D_f$, is not longer true.

From Proposition 8, Theorem 10 and Theorem 22 we get the following

Corollary 24. *Let $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions satisfying the following conditions:*

(i) *there exist $t_0 \in [0,1]$ and $u_0 \in \mathbb{R}$ such that $f(t_0) \neq 0$ and $g(u_0) \neq 0$;*

(ii) *$f(t) = 0$ at every point $t \in [0,1]$ of continuity of f.*

Then the nonautonomous superposition operator F, generated by the function $(t,u) \mapsto f(t)g(u)$, maps the space BV into itself and is locally bounded if and only if:

(a) *$f \in BV$;*

(b) *g is locally bounded.*

Now, we will show that the converse of the claim of Theorem 22 is true under certain additional condition concerning the cardinality of the set D_f of points of discontinuity of f.

Theorem 25. *Let $f\colon [0,1] \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions satisfying the following conditions:*

(i) *there exists $u_0 \in \mathbb{R}$ such that $g(u_0) \neq 0$;*

(ii) *the set D_f is denumerable and $f(t) = 0$ at every point $t \in [0,1]$ of continuity of f.*

Then the nonautonomous superposition operator F, generated by the function $(t,u) \mapsto f(t)g(u)$, maps the space BV into itself if and only if:

(a) *$f \in BV$;*

(b) *g is locally bounded.*

3 Boundedness: the paper II

In the case of autonomous superposition operators the problems concerning the so-called *acting conditions* and local boundedness were solved by M. Josephy who in 1981 established the following

Theorem 26 ([22]). *Suppose that F is an autonomous superposition operator generated by a function $f\colon \mathbb{R} \to \mathbb{R}$. The superposition operator F maps the space BV into itself if and only if the function f satisfies a local Lipschitz condition, that is, for every $r > 0$ there exists a number $L_r \geq 0$ such that $|f(u) - f(w)| \leq L_r|u - w|$, whenever $u, w \in [-r, r]$.*

The main goal of the paper II was to prove that if the superposition operator F maps the space BV into itself, then F is automatically locally bounded, provided that its generator is a locally bounded function. Let us add that the local boundedness of the generator is a necessary condition for the local boundedness of the superposition operators (see paper I, Proposition 2).

Before we proceed further, let us briefly explain the idea behind our approach. The key observation is the following: if the superposition operator maps the space BV into itself, but is not locally bounded, then it is possible to 'transfer' (and then 'localize') its undesired properties to the generator f, that is, it is possible to find a point $(t^*, u^*) \in [0,1] \times \mathbb{R}$ and a sequence $(x_q)_{q\in\mathbb{N}}$ of functions of uniformly bounded variation such that the graphs of the functions are eventually contained in an arbitrary open neighborhood of the point (t^*, u^*) and the corresponding variation sums of the superposition of f and x_q grow to infinity (see Theorem 30). Next, we can show that the functions x_q can be 'redefined' to make their variation on a certain interval around t^* arbitrary small. In the final step it 'suffices' to glue the modified functions together in order to get a function of bounded variation which after superposition with f does not belong to BV (see Theorem 31).

The main result of the paper, Theorem 31, which provides the answer to the question concerning local boundedness of superposition operators, is essential to the theory of such operators in BV-spaces.

3.1 Introductory results

In this section there are presented several technical results that were needed in the sequel to 'translate' the properties of a nonautonomous superposition operator to the properties of its generator. Throughout the section we assume that $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a locally bounded function and that F is a nonautonomous superposition operator generated by f.

Lemma 27. *Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ and suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ of real-valued functions defined on $[0,1]$ is uniformly bounded and such that $\|F(x_n)\|_{BV} \geq n$ for $n \in \mathbb{N}$. Then there exists $t_0 \in [0,1]$ such that for each $\varepsilon > 0$*

$$\sup_{n\in\mathbb{N}} \bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(x_n) = +\infty. \tag{6}$$

Lemma 28. *Let $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ and suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ of real-valued functions defined on $[0,1]$ is uniformly bounded and such that $\|F(x_n)\|_{BV}\geq n$ for $n\in\mathbb{N}$. Then there exists $t_0\in[0,1]$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that for each $\varepsilon>0$*

$$\lim_{k\to\infty}\bigvee_{l_\varepsilon(t_0)}^{r_\varepsilon(t_0)} F(x_{n_k})=+\infty. \tag{7}$$

Our further considerations were based on the following

Lemma 29. *Let $r>0$ and suppose that $x_n\in B_{BV}(0,r)$ and $\|F(x_n)\|_{BV}\geq n$ for $n\in\mathbb{N}$. Then there exists $t_0\in[0,1]$ such that for any $\delta>0$ and $\varepsilon>0$ there exist a subsequence $(x_{n_q})_{q\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, a point $u_0\in[-r,r]$ and a sequence of finite collections of points $l_\varepsilon(t_0)\leq t_0^q<t_1^q<\ldots<t_{d_q}^q\leq r_\varepsilon(t_0)$, where $q\in\mathbb{N}$, for which the following properties hold: $x_{n_q}(t_i^q)\in[u_0-\delta,u_0+\delta]$ for $i=0,\ldots,d_q$ and*

$$\lim_{q\to\infty}\sum_{i=1}^{d_q}\left|f(t_i^q,x_{n_q}(t_i^q))-f(t_{i-1}^q,x_{n_q}(t_{i-1}^q))\right|=+\infty. \tag{8}$$

The proof of Lemma 29 is similar to the proof of Lemma 2 in the paper I.

The following result gives a necessary and sufficient condition for F to map BV into itself and not to be locally bounded.

Theorem 30. *Let $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ be a function mapping bounded sets into bounded sets which generates a nonautonomous superposition operator F that maps the space BV into itself. The superposition operator F is not locally bounded in BV if and only if there exist a number $r>0$ and a point $(t_0,u_0)\in[0,1]\times[-r,r]$, together with a sequence $(x_q)_{q\in\mathbb{N}}$ of functions belonging to $B_{BV}(0,r)$ such that for any $\varepsilon>0$ and $\delta>0$ there exist a sequence of finite collections of points $l_\varepsilon(t_0)<t_0^q<t_1^q<\ldots<t_{d_q}^q<r_\varepsilon(t_0)$ for which the following properties hold: $x_q(t_i^q)\in[u_0-\delta,u_0+\delta]$ for $i=0,1,\ldots,d_q$ and all q sufficiently large, and*

$$\lim_{q\to\infty}\sum_{i=1}^{d_q}\left|f(t_i^q,x_q(t_i^q))-f(t_{i-1}^q,x_q(t_{i-1}^q))\right|=+\infty.$$

3.2 Boundedness of nonautonomuos operators

By Proposition 8, if $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ does not map bounded sets into bounded sets, then there is no hope for the superposition operator F, generated by f, to map bounded subsets of BV into bounded subsets of BV. However, if f maps bounded sets into bounded sets we have the following

Theorem 31. *Let $f\colon[0,1]\times\mathbb{R}\to\mathbb{R}$ be a function mapping bounded sets into bounded ones which generates the nonautonomous superposition operator F. If the superposition operator F maps the space BV into itself, then it is automatically locally bounded.*

From Theorem 31 we get the next

Corollary 32. *Let $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ be a function mapping bounded sets into bounded ones which generates the nonautonomous superposition operator F. If there exists a pointwise bounded sequence $(x_n)_{n\in\mathbb{N}}$ of BV-functions with $\sup_{n\in\mathbb{N}}\bigvee_0^1 x_n < +\infty$ such that $\lim_{n\to\infty}\bigvee_0^1 F(x_n)$ $= +\infty$, then the superposition operator does not map the space BV into itself.*

Thanks to Theorem 31 we can refine the main result of the paper I, Theorem 11, concerning the necessary and sufficient conditions for the inclusion $F(BV)\subset BV$.

Theorem 33. *Let $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ be a function mapping bounded sets into bounded ones which generates the nonautonomous superposition operator F. Then the following conditions are equivalent:*

(i) *the nonautonomous superposition operator F maps the space BV into itself;*

(ii) *for every $r>0$ there exists a constant $M_r>0$ such that for every $k\in\mathbb{N}$, every finite partition $0=t_0<\ldots<t_k=1$ of the interval $[0,1]$ and every finite sequence $u_0,u_1,\ldots,u_k\in[-r,r]$ with $\sum_{i=1}^k|u_i-u_{i-1}|\le r$, the following inequalities hold*

$$\sum_{i=1}^{k}\bigl|f(t_i,u_i)-f(t_{i-1},u_i)\bigr|\le M_r \quad \text{and} \quad \sum_{i=1}^{k}\bigl|f(t_{i-1},u_i)-f(t_{i-1},u_{i-1})\bigr|\le M_r.$$

4 Continuity: the paper III

In the paper [9] it has been proved in a simple manner that if a generator function $f : \mathbb{R} \to \mathbb{R}$ is a sum of power series centered at 0 with infinite radius of convergence, then the autonomous operator F, generated by f, is continuous. Further, in the same paper it has been shown that if we assume that f is of C^1-class, then the autonomous superposition operator generated by f is also continuous. But even more is known in the case of an autonomous operator: the assumption that generator function is of C^1-class implies that the autonomous operator generated by f is uniformly continuous on bounded subsets of BV (see [16, Corollary 6.64]) and if we assume that the generator f is only (locally) lipschitzian, then the autonomous operator may not be uniformly continuous on bounded subsets of BV (see [16, Proposition 6.66]). In the paper [9], it has also been mentioned that Morse's article [28] contains some theorems on continuity of a certain class of operators. It seems that till the publication of the article [9], the paper by Morse had been forgotten for a long time. As the authors of [9] notice, Morse's Theorem 7.1 easily implies that if $f : \mathbb{R} \to \mathbb{R}$ is a locally lipschitzian function, then the autonomous superposition operator F, generated by f, is continuous. Therefore, in view of the main result from the paper [22], for a function $f : \mathbb{R} \to \mathbb{R}$, if the autonomous superposition operator F, generated by f, meets the acting condition $F(BV) \subset BV$, then it is automatically continuous and locally bounded. Let us add, that Morse's result from 1937 implies that if $f : [0,1] \times \mathbb{R}$ is of C^1-class, then the nonautonomous superposition operator F, generated by f, is continuous on the space BV, provided that $F(BV) \subset BV$. As has been observed in [9], Morse's assumptions imply that the generating function is continuous. However, due to the fact that there are continuous superposition operators acting in BV spaces that are generated by discontinuous functions [9, Remark 5], the question of the continuity of a nonautonomous superposition operator still remained open in a general setting.

In this paper we were going to give new proofs of the continuity of an autonomous superposition operator for a lipschitzian generator, of a nonautonomous superposition operator for a C^1-class generator, and we presented necessary and sufficient conditions for the continuity of nonlinear superposition operators in a general setting. In view of the paper [28], it is reasonable to ask why to prove the continuity of superposition operators in the autonomous case for a lipschitzian generator and the continuity of the superposition operator in the nonautonomous case if the generator is continuously differentiable. Basically, there are three reasons for doing that: 1) as remarked in [9], Morse's proof is about 30 pages long and our approach is significantly more concise; 2) Morse's proof uses much more mathematical concepts than our relatively straightforward proofs; we think it makes sense to present (relatively) easy accessible proofs of results that are essential for the theory; 3) we additionally proved that if a generator is of C^1-class, then the superposition operator is uniformly continuous on bounded subsets of BV - this a new result and it is not implied by Morse's Theorem 7.1. As we mentioned, the continuity of an autonomous operator is automatic if the superposition operator maps the space BV into itself. The situation is different in the case of the nonautonomous superposition operators. As we know, it is true that if the generator f is of C^1-class, then the nonautonomous superposition operator generated by f is necessarily continuous. However, if we weaken the assumption of differentiability of the generator f and one "only" assume that f is a locally lipschitzian function, then it may happen that the nonautonomous operator F, generated by f, is not continuous - this is presented by means of a counterexample to follow. The last result proved in the paper describes necessary and sufficient conditions for the continuity of a nonautonomous superposition operator in a general setting in which there are no restrictions on the generator f with the exception that the nonautonomous superposition operator F, generated by f, maps the space BV into itself. It shows up that these conditions take on a form of inequalities resembling the acting conditions presented in the article I.

4.1 The nonautonomous case

The following simple

Lemma 34. *Let the superposition operator F be generated by a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R} \to \mathbb{R}$. Suppose that, for each $\varepsilon > 0$, there exist $\delta > 0$ and functions $\overline{x}, \overline{y} \in BV$ such that $\|\overline{x} - \overline{y}\|_{BV} < \delta$ and $\|F(\overline{x}) - F(\overline{y})\|_{BV} > \varepsilon$. Then, there exist two continuous piecewise linear functions x, y satisfying the inequalities: $\|x\|_{BV} \leq \|\overline{x}\|_{BV}$, $\|y\|_{BV} \leq \|\overline{y}\|_{BV}$, $\|x - y\|_{BV} < \delta$ and $\|F(x) - F(y)\|_{BV} > \varepsilon$. Moreover, there exists a finite partition $0 = t_0 < t_1 < \ldots < t_k = 1$, $k \in \mathbb{N}$, of $[0,1]$, for which $x(t_i) = \overline{x}(t_i)$, $y(t_i) = \overline{y}(t_i)$, and $x(t) = a_i^x t + b_i^x$, $y(t) = a_i^y t + b_i^y$, $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, k$, where a_i^x, a_i^y, b_i^x, b_i^y are some fixed real numbers, $i = 1, \ldots, k$.*

turned out to be crucial for proofs of results presented in this and the following section. Its first consequence was

Theorem 35. *Let $f \in C^1((-a, 1+a) \times \mathbb{R}, \mathbb{R})$, $a > 0$. The nonautonomous superposition operator F, generated by f, is uniformly continuous on bounded subsets of the space BV.*

The below example reveals that the assumption "f is of C^1-class" is reasonable.

Example 36. Let $g : [0,1] \times [1/2, 1] \to \mathbb{R}$ be defined by: for $(t, x) \in [0,1] \times [1/2, 1]$

$$g(t,x) := \begin{cases} t & \text{if } t \leq 1/2,\ x \geq 2t, \\ -t + x & \text{if } t \leq 1/2, x \leq 2t, \\ t + x - 1 & \text{if } t \geq 1/2,\ x \leq 2 - 2t, \\ 1 - t & \text{if } t \geq 1/2,\ x \geq 2 - 2t. \end{cases} \tag{9}$$

The function g is lipschitzian, with a Lipschitz constant 2. Let now $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by: for $(t, x) \in [0,1] \times \mathbb{R}$

$$f(t,x) := \begin{cases} 1/2 - |t - 1/2| & \text{if } x \geq 1, \\ 0 & \text{if } x \leq 0, \\ \frac{1}{2^q} g\left(2^q(t - \frac{1}{2^q}\left\lfloor \frac{t}{1/2^q} \right\rfloor), 2^q x\right) & \text{if } x \in [1/2^{q+1}, 1/2^q],\ q \in \mathbb{N}_0. \end{cases} \tag{10}$$

It is not difficult, but a bit tedious, to check that the f is a lipschitzian function (see also Figure 1 below). Now, let us observe that the nonautonomous superposition operator F, generated by f, maps the space BV into itself - this a simple consequence of the fact that f is lipschitzian.
We have $F(\theta)(t) = f(t, 0) = 0$, $t \in [0,1]$, and thus $||F(\theta)||_{BV} = 0$. Let now $x^q(t) :=$

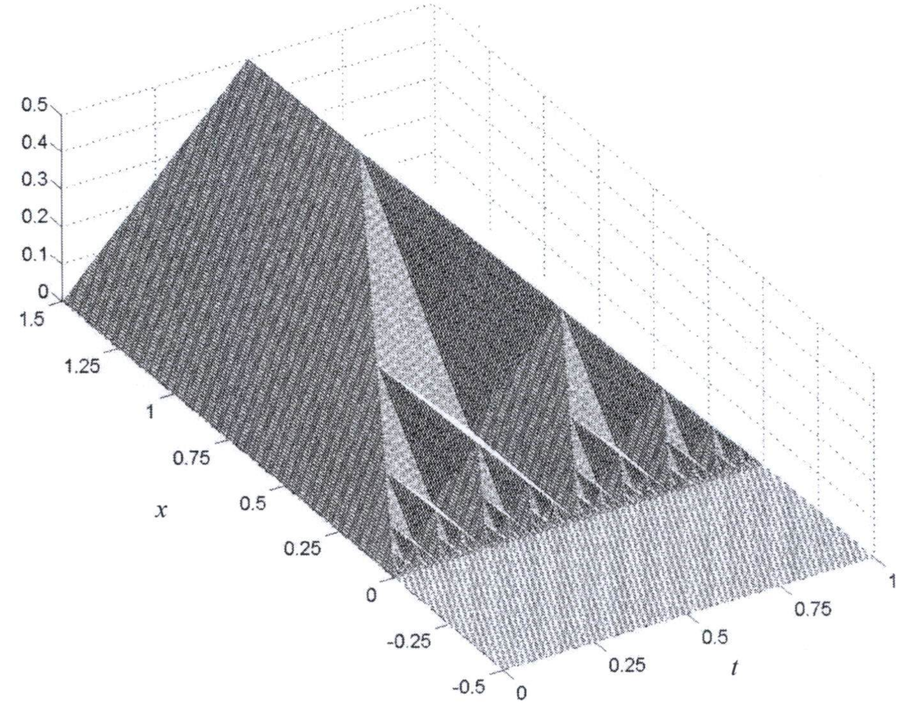

Figure 1: A part of the graph of the function f defined in Example 4.

$1/2^q$, $t \in [0,1]$, $q \in \mathbb{N}_0$. It holds $||x^q - \theta||_{BV} = ||x^q||_{BV} = 1/2^q$, $q \in \mathbb{N}_0$, and hence $x^q \to \theta$ in BV as $q \to +\infty$. At the same time, for every $q \in \mathbb{N}_0$, we have $x^q(0) = 0$ and

$$\bigvee_0^1 F(x^q) = \bigvee_0^1 f(\cdot, 1/2^q) = 1.$$

It is clear that it cannot be $\lim_{q\to+\infty} F(x^q) = F(\theta)$, though $\lim_{q\to+\infty} x^q = \theta$. Let us observe that the generator f constructed in the example is lipschitzian with respect to both variables. Nevertheless, it is not sufficient for the continuity of the operator F, generated by f.

4.2 The autonomous case

Let us now assume that the superposition operator F, generated by a function $f : \mathbb{R} \to \mathbb{R}$, maps the space BV into itself. It is well-known that in this case f is locally lipschitzian [22]. It is also true, that the lipschitz continuity of a generator guarantees the continuity of the operator it generates, which is presented in

Theorem 37. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally lipschitzian function. The autonomous superposition operator F, generated by f, is continuous.*

4.3 Necessary and sufficient conditions for continuity in general setting

The main result of the paper III is the following

Theorem 38. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a function such that the superposition operator F, generated by f, maps the space BV into itself. Let $x \in BV$ be fixed. The following conditions are equivalent:*

(38.1) *the operator F is continuous at x,*

(38.2) *for each $t \in [0,1]$, the function $\mathbb{R} \ni u \mapsto f(t,u) - f(t,x(t))$ is continuous at $u = x(t)$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $k \in \mathbb{N}$, every partition $0 = t_0 < \ldots < t_k = 1$ of the interval $[0,1]$, and every finite sequence $u_0, u_1, \ldots, u_k \in [-\delta, \delta]$ with $\sum_{i=1}^{k} |u_i - u_{i-1}| \leq \delta$, we have*

$$\sum_{i=1}^{k} |[f(t_i, u_i + x_i) - f(t_{i-1}, u_i + x_{i-1})] - [f(t_i, x_i) - f(t_{i-1}, x_{i-1})]| \leq \varepsilon, \text{ and}$$

$$\sum_{i=1}^{k} |f(t_{i-1}, u_i + x_{i-1}) - f(t_{i-1}, u_{i-1} + x_{i-1})| \leq \varepsilon,$$

where $x_i := x(t_i)$, $i = 0, 1, \ldots, k$.

PIOTR MAĆKOWIAK

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